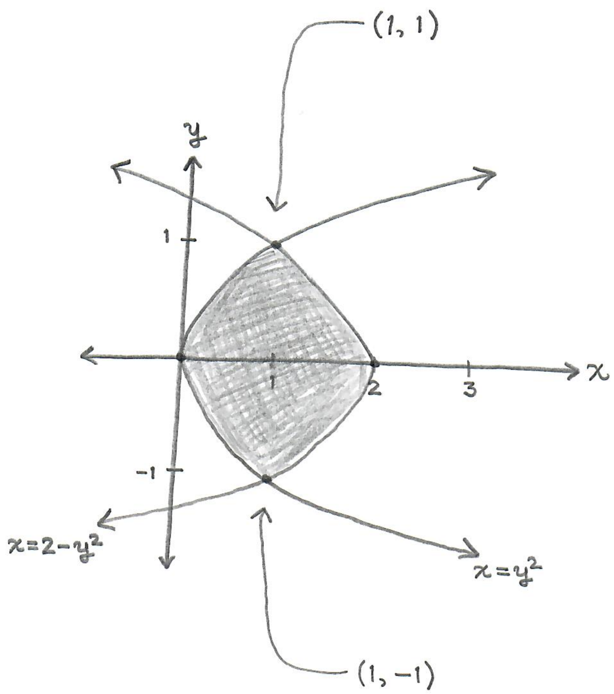
MATH-172: EXAM 1

1. Sketch the region enclosed by the given curves and find its area:

$$x = 2 - y^2 \qquad\qquad x = y^2$$



$$\text{area} = \int_{-1}^{1} \left[(2-y^2)-(y^2)\right] dy$$

$$= \int_{-1}^{1} (2-y^2-y^2)\, dy$$

$$= \int_{-1}^{1} (2-2y^2)\, dy$$

$$= \left(2y - \frac{2}{3}y^3\right)\Big|_{-1}^{1}$$

$$= \left[2(1)-\frac{2}{3}(1)^3\right] - \left[2(-1)-\frac{2}{3}(-1)^3\right]$$

$$= \left[2-\frac{2}{3}\right] - \left[-2-\left(-\frac{2}{3}\right)\right]$$

$$= 2-\frac{2}{3}+2-\frac{2}{3}$$

$$= 4-\frac{4}{3}$$

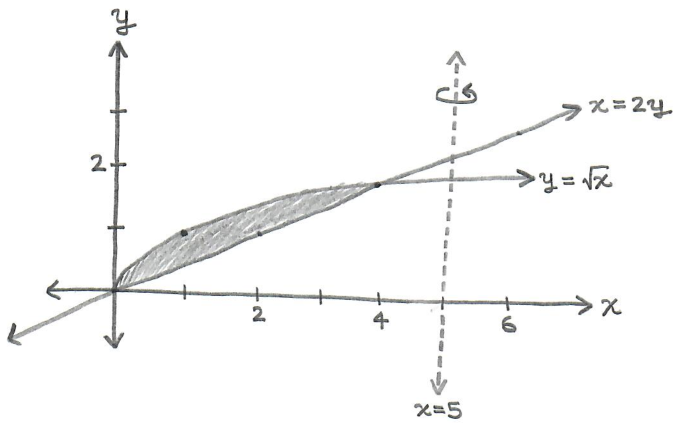
$$= \boxed{\frac{8}{3}}$$

2. Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about $x = 5$:

$$x = 2y \qquad\qquad y = \sqrt{x}$$

$$\downarrow$$

$$y = \tfrac{1}{2}x$$



INTERSECTION POINTS

$$\tfrac{1}{2}x = \sqrt{x}$$

$$\tfrac{1}{4}x^2 = x$$

$$\tfrac{1}{4}x^2 - x = 0$$

$$x\left(\tfrac{1}{4}x - 1\right) = 0$$

$x = 0$ $\qquad$ $x = 4$

$$\text{volume} = \int_0^4 2\pi(5-x)\left[(\sqrt{x}) - \left(\tfrac{1}{2}x\right)\right] dx$$

$$= 2\pi \int_0^4 (5-x)\left(\sqrt{x} - \tfrac{1}{2}x\right) dx$$

$$= 2\pi \int_0^4 \left(5\sqrt{x} - \tfrac{5}{2}x - x\sqrt{x} + \tfrac{1}{2}x^2\right) dx$$

$$= 2\pi \int_0^4 \left(5x^{1/2} - \tfrac{5}{2}x - x^{3/2} + \tfrac{1}{2}x^2\right) dx$$

$$= 2\pi\left(5\cdot\tfrac{2}{3}x^{3/2} - \tfrac{5}{2}\cdot\tfrac{1}{2}x^2 - \tfrac{2}{5}x^{5/2} + \tfrac{1}{2}\cdot\tfrac{1}{3}x^3\right)\Big|_0^4$$

$$= 2\pi\left(\tfrac{10}{3}x^{3/2} - \tfrac{5}{4}x^2 - \tfrac{2}{5}x^{5/2} + \tfrac{1}{6}x^3\right)\Big|_0^4$$

$$= 2\pi\left[\left(\tfrac{10}{3}(4)^{3/2} - \tfrac{5}{4}(4)^2 - \tfrac{2}{5}(4)^{5/2} + \tfrac{1}{6}(4)^3\right) - (0)\right]$$

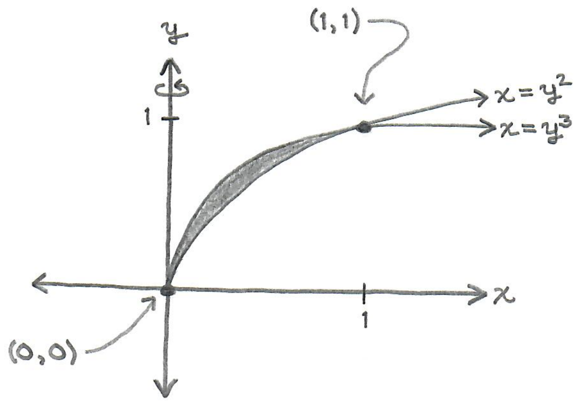
$$= 2\pi\left(\tfrac{80}{3} - 20 - \tfrac{64}{5} + \tfrac{64}{6}\right)$$

$$= 2\pi\left(\tfrac{68}{15}\right)$$

$$= \boxed{\tfrac{136\pi}{15}}$$

3. Use the washer method to find the volume of the solid obtained by rotating the region bounded by the given curves about the y-axis:

$$x = y^2 \qquad\qquad x = y^3$$



$$\text{Volume} = \int_0^1 \pi\left[(y^2)^2 - (y^3)^2\right] dy$$

$$= \pi \int_0^1 (y^4 - y^6)\, dy$$

$$= \pi\left(\tfrac{1}{5}y^5 - \tfrac{1}{7}y^7\right)\Big|_0^1$$

$$= \pi\left[\left(\tfrac{1}{5}(1)^5 - \tfrac{1}{7}(1)^7\right) - (0)\right]$$

$$= \pi\left(\tfrac{1}{5} - \tfrac{1}{7}\right)$$

$$= \pi\left(\tfrac{2}{35}\right)$$

$$= \boxed{\frac{2\pi}{35}}$$

4. Suppose you have the function $f(x) = \sqrt{9-x}$ on the interval $[0, 5]$.

(a) Find the average value of f on the given interval.

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{5-0}\int_0^5 \sqrt{9-x}\,dx = \frac{1}{5}\int_0^5 \sqrt{9-x}\,dx$$

LET $u = 9-x$

$\downarrow$

$du = (-1)\,dx$

$dx = (-1)\,du$

$u_5 = 9-(5) = 4$

$u_0 = 9-(0) = 9$

$$= \frac{1}{5}\int_9^4 \sqrt{u}\cdot(-1)\,du = -\frac{1}{5}\int_9^4 \sqrt{u}\,du$$

$$= \frac{1}{5}\int_4^9 u^{1/2}\,du = \frac{1}{5}\left(\frac{2}{3}u^{3/2}\right)\Big|_4^9 = \frac{1}{5}\left[\frac{2}{3}(9)^{3/2} - \frac{2}{3}(4)^{3/2}\right]$$

$$= \frac{1}{5}\left(18 - \frac{16}{3}\right) = \frac{1}{5}\left(\frac{38}{3}\right) = \boxed{\frac{38}{15}}$$

(b) Find c in the given interval such that $f_{\text{avg}} = f(c)$.

$\downarrow$

$$\frac{38}{15} = \sqrt{9-c}$$

$$\frac{1444}{225} = 9-c$$

$$\frac{1444}{225} - 9 = -c$$

$$\frac{1444}{225} - \frac{2025}{225} = -c$$

$$-\frac{581}{225} = -c$$

$\downarrow$

$$\boxed{c = \frac{581}{225}}$$

5. A 10-ft chain weighs 25 lb and hangs from a ceiling. Find the work done in lifting the lower end of the chain to the ceiling so that it is level with the upper end.

force $= F = \dfrac{25 \text{ lb}}{10 \text{ ft}} = 2.5 \text{ lb/ft}$

Only the lower half of the chain moves up to the ceiling, so the distance moved is $d = \frac{1}{2}x$

$$\text{work} = \int_a^b (F \cdot d)\, dx = \int_0^{10} \left(2.5 \cdot \tfrac{1}{2}x\right) dx = \int_0^{10} 1.25x \, dx$$

$$= 1.25 \cdot \tfrac{1}{2}x^2 \Big|_0^{10} = 0.625x^2 \Big|_0^{10} = 0.625\left[(10)^2 - (0)^2\right]$$

$$= 0.625(100) = 62.5$$

$\boxed{\text{The work done in lifting the lower end of the chain to the ceiling is 62.5 ft-lb.}}$